NMSU MATH PROBLEM OF THE WEEK

Solution to Problem 8

Fall 2024

Problem 8

Some people visit a petting zoo. Exactly three visitors pet every animal, exactly one visitor pets every pair of animals, and no visitor pets every animal.

What is the largest possible number of animals in the petting zoo? Justify your answer.

Solution. We will first show that no visitor can pet four or more animals. Indeed, suppose that visitor A pets four animals a, b, c, and d. Then since no visitor pets every animal there is another animal e which A does not pet. But then there must be visitors who pet the pairs of animals $(a, e), (b, e), (c, e)$, and (d, e). Furthermore these are 4 distinct visitors since otherwise they would pet the same pair of animals as visitor A. This is a contradiction since we now have 4 visitors who pet animal e.

We now show that there are at most 7 animals in the petting zoo. Indeed, the maximum number of animals will be when every visitor pets three animals, in which case we have the same number of visitors as animals since three visitors pet every animal. We now note that since exactly one visitor pets every pair of animals, and every visitor pets exactly 3 pairs of animals, we have that the number of visitors is equal to $\frac{1}{3}$ the number of pairs of animals. Let n be the number of visitors and, we calculate:

$$\begin{aligned} \frac{1}{3}\binom{n}{2} &= n, \text{which simplifies to} \\ n^2 - 7n &= 0, \text{and hence} \\ n &= 0, \text{ or } n = 7. \end{aligned}$$

Thus there are at most 7 animals in the zoo. It remains to check that it is possible to have 7 animals in the zoo, for which we write out an explicit example. Number the animals 1-7, and have the 7 visitors pet the following triples of animals: $(1, 2, 3), (1, 4, 5), (1, 6, 7), (2, 4, 6), (2, 5, 7), (3, 4, 7)$, and $(3, 5, 6)$. It is easy to check that three visitors pet every animal, one visitor pets every pair of animals, and no visitor pets every animal.